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Rolling among G_2 vacua

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Abstract: We consider topology-changing transitions between 7-manifolds of holonomy G_2 constructed as a quotient of $CY \times S^1$ by an antiholomorphic involution. We classify involutions for Complete Intersection CY threefolds, focussing primarily on cases without fixed points. The ordinary conifold transition between CY threefolds descends to a transition between G_2 manifolds, corresponding in the $\mathcal{N} = 1$ effective theory incorporating the light black hole states either to a change of branch in the scalar potential or to a Higgs mechanism. A simple example of conifold transition with a fixed nodal point is also discussed. As a spin-off, we obtain examples of G_2 manifolds with the same value for the sum of Betti numbers $b_2 + b_3$, and hence potential candidates for mirror manifolds.

Keywords: M-Theory, Superstring Vacua, Black Holes in String Theory.

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1. Introduction

One of the important discoveries of the past few years in string theory is that a vast number of consistent string models are not disconnected, but actually different vacua related by smooth deformations or more radical phase transitions. This view has emerged from a better understanding of the non-perturbative spectrum of string theories and of their non-perturbative dualities. In particular, for type II string theories with $\mathcal{N} = 2$ supersymmetry in four dimensions, it has been found that a wide class of Calabi-Yau (CY) compactifications were related by smooth topology-changing "conifold" transitions [1], whereby a two-cycle shrinks to zero size and reappears as a three-cycle (see also [2]–[4]). While this transition was known at the mathematical level [5], string theory gives a smooth representation of this transition by providing the low-energy degrees of freedom that resolve the singularity on the conformal field theory moduli space: namely, the Ramond-Ramond charged black holes that become massless at the singularity [6, 7]. The purpose of this work is to study such topology-changing transitions in a $\mathcal{N}=1$ setting.

$\mathcal{N}=1$ vacua in four dimensions can be constructed in many ways. The heterotic string compactified on a CY threefold, possibly in the presence of 5-brane sources, has been intensively studied but is complicated by the fact that a superpotential may be generated already at tree-level by worldsheet instantons [8, 9]. Type I strings or orientifold of type II on CY threefolds are another option, and conifold transitions were in particular considered in type I' string theory in [10]. This can also be reformulated geometrically as compactifications of F-theory on a CY fourfold [11], or by considering space-filling branes wrapped on supersymmetric cycles in type II compactifications on CY threefolds [12]–[14]. In this work, we consider another geometric realization, namely M-theory compactified on 7-manifolds of exceptional holonomy G_2. Using the invariance under Peccei-Quinn-type symmetries, one can argue that in this case the superpotential may arise only at the non-perturbative level from membrane instantons [15, 16]. Joyce has proposed a relatively simple construction of G_2 manifolds [17, 18], as a quotient

$$\mathcal{G}=(\mathcal{C}\times S^1)/\mathbb{Z}_2 \tag{1.1}$$

of the product of a CY threefold $\mathcal{C}$ and a circle S^1 by an involution $\sigma=w\mathcal{I}$ acting as an inversion $\mathcal{I}$: $x^{10}\to -x^{10}$ on S^1 and as an antiholomorphic involution w on $\mathcal{C}$ such that $w^*(J)=-J$ and $w^*(\Omega)=e^{i\theta}\bar{\Omega}$, where J and Ω are the Kähler form and holomorphic three-form, while θ is a real constant. The main goal of this paper is to study topology-changing transitions between G_2 manifolds of the form (1.1) resulting from conifold transitions in $\mathcal{C}$. We shall focus on the simplest (abelian) type of transition, but our discussion could easily be generalized to more complicated non-abelian transitions [19]–[22]. Compactifications on singular G_2 manifolds have also been considered from the point of view of geometric engineering in[1] [23]. We shall also disregard the superpotential that might be generated on either side of the transition by instantons. Such a superpotential may lift part or all of the branches of the moduli space on either side, and in particular drive the theory to the conifold point. At any rate, it does not prevent a continuous transition between the remaining vacua in *configuration* space, which is a necessary condition for the existence of tunneling processes between $\mathcal{N}=1$ vacua.

Besides extremal transitions, CY manifolds can also be related by mirror symmetry. This is not a continuous transition between different CY's proper, but rather

[1] As this work was finalized, a preprint appeared [39] which discusses non-abelian singularities in local models of G_2 manifolds.

a smooth cross-over between different geometric descriptions of a same CFT in different regime of moduli space. Mirror symmetry has also been conjectured to hold in the context of G_2 manifolds [24, 25], although much less is known, for lack of a precise understanding of the CFT. Although this is somewhat peripheral to the focus of the present work, we shall also exhibit various examples of G_2 manifolds, whose Betti numbers satisfy $b_2 + b_3 =$ constant. This relation is a necessary condition for G_2 manifolds to be mirror [24], and it would be interesting to investigate whether the CFT's underlying these examples are indeed equivalent.

The plan of this work is as follows. We start in section 2 by recalling some background about G_2 manifolds, discuss Joyce's construction and classify possible antiholomorphic involutions. Section 3 is a short review of conifold transitions in Complete Intersection Calabi-Yau (CICY) manifolds on which we focus in the following. In section 4, we discuss transitions between G_2 manifolds triggered by a conifold transition in the underlying CICY, both from the mathematical and physical point of view. For the latter, we show that conifold transitions correspond to either a change of scalar branch in the $\mathcal{N} = 1$ moduli space, or to a standard Higgs effect. In section 5, we briefly discuss the case where the involution has a non-empty fixed point set, and in particular when it contains the nodal points of the conifold.

2. G_2 manifolds as Calabi-Yau orbifolds

2.1 General facts about G_2 manifolds

Seven-manifolds of exceptional holonomy $G_2 \subset \mathrm{SO}(7)$ have one covariantly constant spinor θ, as apparent from the branching rule $\mathbf{8} = \mathbf{7} \oplus \mathbf{1}$ of the spinor representation of $SO(7)$. Equivalently, there is one covariantly constant three-form ϕ, closed and co-closed [26]. Locally, one may choose an orthogonal frame e_i in which

$$\phi = e_{127} + e_{136} + e_{145} + e_{235} + e_{426} + e_{347} + e_{567}\,, \quad e_{ijk} = e_i \wedge e_j \wedge e_k\,, \tag{2.1}$$

where we recognize on the r.h.s. the structure constants of the unit octonions. Compact G_2 manifolds provide $\mathcal{N} = 1$ supersymmetric backgrounds for classical eleven-dimensional supergravity. The massless spectrum follows from simple homological considerations [27]. The deformations of the three-form ϕ yield b_3 real moduli, which combine with the flux of the 3-form C and the modes of the gravitino into b_3 $\mathcal{N} = 1$ chiral multiplets. In addition, the reduction of the three-form on the b_2 2-cycles yields b_2 gauge fields, which together with the reduction of the gravitino make up b_2 $\mathcal{N} = 1$ vector multiplets. The Kähler potential and gauge kinetic term are simply obtained from the volume V of the manifold and intersection matrix respectively, while the superpotential vanishes in the classical supergravity approximation. By the usual arguments of holomorphy and Peccei-Quinn symmetry, it remains zero to all orders in $1/V$, but may be generated by membrane instantons [16].

2.2 Joyce's construction of G_2 manifold

The first examples of compact G_2 manifolds have been constructed by orbifold constructions T^7/Γ, where Γ is a discrete group contained in a G_2 subgroup of SO(7) [17, 18]. The fixed points singularities can be resolved by appropriately gluing in Eguchi-Hanson spaces with SU(2) holonomy, so that the total holonomy lies in all of G_2. This construction can be repeated by orbifolding other compact 7-manifolds with reduced holonomy such as $K_3 \times T^3$ or, more generally, $\mathcal{C} \times S^1$ for a CY threefold $\mathcal{C}$. One thus considers quotients

$$\mathcal{G} = (\mathcal{C} \times S^1)/\sigma \,, \tag{2.2}$$

where $\sigma = w\mathcal{I}$ is an involution acting as $\mathcal{I}$: $x \to -x$ on S^1 and antiholomorphically on $\mathcal{C}$. σ must in addition be an isometry, so that $w^*(J) = -J$ and $w^*(\Omega) = e^{i\theta}\bar{\Omega}$. The closed and co-closed four-form

$$\phi = J \wedge dx + \mathrm{Re}(e^{-i\theta/2}\Omega) \tag{2.3}$$

is invariant under σ, and provides the quotient $(\mathcal{C} \times S^1)/\mathbb{Z}_2$ with a G_2 structure.

In general, the involution w may have a non-empty fixed point set Σ on $\mathcal{C}$, which is then a compact special Lagrangian 3-cycle [28]. In that case, one must in addition resolve the singularity of the quotient, by gluing in an Eguchi-Hanson space in the space transverse to Σ. It appears necessary to have $b_1(\Sigma) > 0$ in order for a resolution to exist. This is not surprising, since this is also the number of deformations of the special Lagrangian 3-cycle Σ [29]. In the following, we concentrate on orbifolds without fixed points, but we will return to the problem of fixed points in section 5.

The Betti numbers of the manifold in (2.2) can be counted as follows. Let us denote h_{11}, h_{12} the Hodge numbers of the CY and $h_{11}^{\pm}$ the number of even (odd) two-forms. The number of invariant two-forms on $(\mathcal{C} \times S^1)/w\mathcal{I}$ is then simply h_{11}^{+}, while three-forms are obtained by wedging the h_{11}^{-} odd two-forms on $\mathcal{C}$ with dx. Moreover, the real parts of the three-form in $H^{1,2}(\mathcal{C}) \oplus H^{2,1}(\mathcal{C})$ and $H^{0,3}(\mathcal{C}) \oplus H^{3,0}(\mathcal{C})$ are also invariant. The untwisted Betti numbers of $\mathcal{G}$ are therefore

$$b_2 = h_{11}^{+} \quad \text{and} \quad b_3 = 1 + h_{11}^{-} + h_{12} \,. \tag{2.4}$$

In the non-freely acting case, one has to add the contribution of the desingularization of the fixed points, as we discuss in section 5.

For simplicity, we will restrict our attention to G_2 manifolds constructed out of Complete Intersection CY (CICY) threefolds [30, 31], *i.e.* defined by a set of homogeneous equations in a product of projective spaces $\mathcal{P} = \prod_{k=1}^{r} \mathbb{CP}^{n_k}$. The full moduli space to which such a manifold belongs is specified by a configuration matrix of the form

$$\big[\, V \big\| D \,\big]_{\chi}^{h_{11},h_{12}} \,, \tag{2.5}$$

where V is a column r-vector, whose entries are the dimensions of the r embedding $\mathbb{CP}^{n_k}$ factors, and D is a $r \times h$ matrix, where $h = \sum_{k=1}^{r} n_k - 3$ is the number of homogeneous polynomial constraints on the projective coordinates. The entry $D_{k,l}$ is the degree of the l-th constraint in the homogeneous coordinates of $\mathbb{CP}^{n_k}$. $\chi = 2(h_{11} - h_{12})$ is the Euler characteristic, which is easily computed from the matrix (2.5) [32, 31]. In order for the manifold to be Ricci flat, we must have $\sum_{l=1}^{h} D_{kl} = n_k + 1$, for all $k = 1, \ldots, r$. A particular set of antiholomorphic involutions is then obtained by restricting antiholomorphic involutions of the projective space $\mathcal{P}$. Requiring this involution to be an isometry of the CY will restrict the allowed coefficients of the homogeneous equations.

2.3 Antiholomorphic involutions of projective spaces

Let us start by classifying the antiholomorphic involutions w of a single projective space $\mathbb{CP}^n$. We represent them by the matrix M such that $z_i \to M_{ij}\bar{z}_j$. M and ρM define the same involution of $\mathbb{CP}^n$ for any $\rho \in \mathbb{C}^*$, so that we may choose $\det M = 1$. These involutions have to be classified up to a holomorphic change of basis $z_i \to U_{ij} z_j$, which amounts to $M \to U^{-1} M U^*$, where $U \in Gl(n+1, \mathbb{C})$. Requiring w to square to 1 projectively imposes $MM^* = \lambda I$. Taking the trace and determinant of this equation, we see that λ is real and $\lambda^{n+1} = 1$. For n even, this forces $\lambda = 1$, while n odd allows for the two possibilities $\lambda = \pm 1$. Requiring furthermore w to be an isometry (i.e. preserving the Fubini-Study metric of $\mathbb{CP}^n$) imposes $MM^\dagger = \mu I$, where μ is fixed to 1 by the previous choices. Combining this equation with $MM^* = \lambda$ implies that M is symmetric for $\lambda = 1$ and antisymmetric for $\lambda = -1$. In either cases, the real and imaginary parts of M commute again due to $MM^* = \lambda I$. They can therefore be simultaneously brought into a diagonal form for $\lambda = 1$ or antisymmetric diagonal form for $\lambda = -1$ by a real orthogonal rotation, hence an allowed holomorphic change of basis. Finally, the phase of the coefficients can be reabsorbed by an holomorphic change of basis. Altogether, we have thus found two distinct antiholomorphic involutions,

$$\begin{aligned} A: &\ (z_1, z_2, \ldots, z_{n+1}) \to (\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_{n+1}) \\ B: &\ (z_1, z_2, \ldots, z_n, z_{n+1}) \to (-\bar{z}_2, \bar{z}_1, \ldots, -\bar{z}_{n+1}, \bar{z}_n)\,. \end{aligned} \tag{2.6}$$

The two cases correspond to different values of λ and cannot be combined in the same projective factor $\mathbb{CP}^n$ without spoiling the involution property. In particular, the involution B is defined for n odd only, and exchanges the projective coordinates by pairs. These two involutions have very different properties, since A admits a fixed set $\{z_i = \bar{z}_i\}$, while B acts freely. For $n = 1$, these are the reflection along the equator and the antipodal map of the sphere S^2, respectively.

Finally, we may consider antiholomorphic involutions that mix different factors in $\prod_{k=1}^{r} \mathbb{CP}^{n_k}$. The involution must commute with the projective actions, so that the only possibility is to exchange two identical projective factors,

$$C: \quad (\{y_i\}\,;\{z_i\}) \to (\{\bar{z}_i\}\,;\{\bar{y}_i\}). \tag{2.7}$$

This involution has a fixed point set $\{y_i = \bar{z}_i\}$, which is the diagonal in $\mathbb{CP}^n \times \mathbb{CP}^n$.

2.4 Antiholomorphic involutions of CICY

Having constructed antiholomorphic involutions of projective spaces A, B, C, we can now combine them to construct G_2 manifolds from CICY 3-folds as in (2.2). We shall denote the resulting manifolds by the configuration matrix of the underlying CY, denoting by $\overline{n}$, $\widehat{n}$, and $\begin{matrix}\overset{\frown}{n}\\ \underset{\smile}{n}\end{matrix}$ the projective spaces in the first column on which the involution acts by A, B or C, respectively. For example, the configuration matrix

$$\left[\begin{array}{c||cccccc} \widehat{7} & 1 & 1 & 1 & 1 & 2 & 2 \\ \overset{\frown}{1} & 1 & 1 & 0 & 0 & 0 & 0 \\ \underset{\smile}{1} & 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right]^{1,54}, \tag{2.8}$$

denotes the family of G_2 manifolds constructed from the CICY with the same configuration matrix, by acting with the involution B on $\mathbb{CP}^7$ and C on $\mathbb{CP}^1 \times \mathbb{CP}^1$ (this example will be treated in detail in section 4.1.3). The superscripts indicate the Betti numbers b_2 and b_3 counting the two-and three-cycles invariant under the involution, respectively. When the involution has fixed points, it is necessary to add the contribution of the singularities after resolution in order to obtain the correct topological invariants.

In order for the projective space involution w to restrict to the CY 3-fold, reality conditions must be enforced on the coefficients of the homogeneous equations. This generically halves the number of allowed complex deformations of the CY. In some cases however, there is simply no choice of coefficients which are preserved by the involution. An example of this is the matrix

$$\left[\begin{array}{c||cccc} \widehat{5} & 3 & 1 & 1 & 1 \\ \overline{1} & 1 & 1 & 0 & 0 \\ \widehat{1} & 0 & 0 & 1 & 1 \end{array}\right]^{0,73} \tag{2.9}$$

for which it is easy to convince oneself that there is no choice of equation of bidegree $(3, 1)$ in the coordinates of $\mathbb{CP}^5 \times \mathbb{CP}^1$ compatible with the involution. It is thus important in practice to check that the projective space involution is compatible with the CICY configuration matrix.

A second important remark is that the topology of the G_2 manifold is not fully specified by the configuration matrix. Instead, the moduli space of G_2 manifolds associated to a given configuration matrix in general splits into several disconnected components whose Betti numbers differ from each other by the contribution of fixed points. This is because the locus of fixed points under a given involution w undergoes transitions in real codimension 1, whereby real roots collide and become imaginary in complex conjugate pairs. In the following, we shall consider cases where the choice of involution ensures that there is no fixed point throughout the CY moduli space, so that one obtains only one G_2 manifold moduli space. We postpone to section 5 the discussion of the more challenging case where this assumption is not valid.

3. A review of conifold transition between CICY's

The aim of this section is to review in some detail some of the mathematical aspects of conifold singularities and transitions between complete intersection CY's [1]. A similar approach will then be taken in the next section in the $\mathcal{N}=1$ case.

3.1 An example

Let us consider a CY manifold $\mathcal{C}_1$ chosen at a generic point of the moduli space $\mathcal{M}^{(1)}$ associated to the configuration matrix

$$\mathcal{M}^{(1)} = \left[\begin{array}{c||ccccc} 7 & 1 & 1 & 2 & 2 & 2 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{array}\right]^{2,58}_{-112} . \tag{3.1}$$

Define x_i $(i=1,\dots,8)$ and y_j $(i=1,2)$ the projective coordinates in $\mathbb{CP}^7$ and $\mathbb{CP}^1$, respectively. The defining equations of the manifold take then the form

$$(\mathcal{S}_1) \left\{ \begin{array}{l} f_1(x,y) := P_{11}(x)y_1 + P_{12}(x)y_2 = 0 \\ f_2(x,y) := P_{21}(x)y_1 + P_{22}(x)y_2 = 0 \\ e_2(x) = e_3(x) = e_4(x) = 0\,, \end{array} \right. \tag{3.2}$$

where the $P_{k,l}$ $(k,l=1,2)$ and e_n $(n=2,3,4)$ are homogeneous polynomials in x_i's of degree one and two, respectively. For generic coefficients, these equations are transverse, which means that $f_i = e_n = 0$ together with $df_1 \wedge df_2 \wedge de_2 \wedge de_3 \wedge de_4 = 0$ has no solution. Changing the complex coefficients appearing in the defining polynomials amounts to changing the complex structure of the manifold[2]. The Kähler moduli on the other hand correspond to the volumes v_7 and v_1 of $\mathbb{CP}^7$ and $\mathbb{CP}^1$, respectively (together with the fluxes of the three-form C on the three-cycles dual to $J_{1,7} \wedge dx^{10}$).

Since y_j $(j=1,2)$ are projective coordinates, in order to have non-vanishing solutions to $f_1 = f_2 = 0$ in (3.2), the matrix of coefficients $P_{k,l}$ must have vanishing determinant

$$e_1^\sharp(x) := P_{11}(x)P_{22}(x) - P_{21}(x)P_{12}(x) = 0\,. \tag{3.3}$$

[2] Actually, there is not in general a one-to-one correspondence between the independent polynomial deformations and the complex structure moduli. See [33, 32] for details.

We may thus dispose of the $\mathbb{CP}^1$ coordinates (y_1, y_2) altogether, and rewrite the system as

$$(\mathcal{S}_0^\sharp) \begin{cases} e_1^\sharp(x) = 0 \\ e_2(x) = e_3(x) = e_4(x) = 0\,, \end{cases} \tag{3.4}$$

where the variables $y_{1,2}$ have been "integrated out". This amounts to having shrunk the $\mathbb{CP}^1$ parameterized by $y_{1,2}$ to zero size (in particular, the Kähler class of the $\mathbb{CP}^1$ has now disappeared). We are then left with a variety $(\mathcal{C}_0^\sharp)$ in

$$\mathcal{M}^{(0)} = \big[\,7\big\|2\ 2\ 2\ 2\,\big]^{1,65}_{-128}\,. \tag{3.5}$$

This operation is called a *determinantal contraction*, and is denoted by

$$\big[\,7\big\|2\ 2\ 2\ 2\,\big]^{1,65}_{-128} \leftarrow \left[\begin{matrix}7\\1\end{matrix}\middle\|\begin{matrix}1&1&2&2&2\\1&1&0&0&0\end{matrix}\right]^{2,58}_{-112}\,, \tag{3.6}$$

(the reversed operation being called *determinantal splitting*). Conversely, at a generic point on (3.5), the matrix

$$\begin{pmatrix} P_{11}(x) & P_{12}(x) \\ P_{21}(x) & P_{22}(x) \end{pmatrix} \tag{3.7}$$

has rank one: it therefore determines a unique projective solution of (3.2), which means that a point in $\mathcal{C}_0^\sharp$ corresponds to a point in $\mathcal{C}_1$. However, when all $P_{k,l}$ vanish, the space (3.4) is singular, and there is a full $\mathbb{CP}^1$-worth of (y_1, y_2) satisfying (3.2). Since $P_{k,l}(x) = 0$ $(k, l = 1, 2)$ and $e_n(x) = 0$ $(n = 2, 3, 4)$ give us 7 conditions for 7 inhomogeneous coordinates in $\mathbb{CP}^7$, this happens at isolated points on $\mathcal{C}_0^\sharp$ known as nodal points. A simple counting shows that there are 8 of them. The manifold $\mathcal{C}_1$ therefore gives a resolution of the singular manifold $\mathcal{C}_0^\sharp$, where $\mathbb{CP}^1$'s are glued at each of the nodes.

There is actually another way to desingularize $\mathcal{C}_0^\sharp$, that is to change the coefficients of the degree two polynomial $e_1^\sharp$. The deformed space is then defined by

$$(\mathcal{S}_0) \begin{cases} e_1(x) := e_1^\sharp(x) - t\varepsilon^2(x) = 0 \\ e_2(x) = \ e_3(x) = e_4(x) = 0\,, \end{cases} \tag{3.8}$$

where t is some sufficiently small but not zero real number and $\varepsilon^2(x)$ is any homoneneous polynomial of degree 2 chosen such that it is non zero at any of the 8 singular points of $\mathcal{C}_0^\sharp$. We have thus *deformed* $\mathcal{C}_0^\sharp$ to a smooth manifold $\mathcal{C}_0$ in $\mathcal{M}^{(0)}$.

The resolution and deformation described above in fact correspond to the two ways of desingularizing the local neighborhood of each node, which is homeomorphic to a real cone over $S^2 \times S^3$. In $\mathcal{C}_1$ the apex of each cone is blown-up into a sphere S^2, while in $\mathcal{C}_0$ the apex is blown up into a sphere S^3. The transition between the two is known as a conifold transition. The change in the Euler characteristic accross the transition between two CY's $\mathcal{C}$ and $\mathcal{C}'$ is then in general simply understood: Since $\chi(S^2) = 2$, while $\chi(S^3) = 0$, we have

$$\frac{1}{2}[\chi(\mathcal{C}) - \chi(\mathcal{C}')] = N\,, \tag{3.9}$$

where N counts the number of nodes at the transition. The change in the Hodge number is more difficult to compute, and will be explained in section 3.3.

3.2 The web of complete intersection CY's

The determinantal splitting illustrated in the example of the previous section can now be repeated successively for each of the degree two equations $e_{2,3,4}$. One then obtains a sequence of transitions $\mathcal{M}^{(0)} \leftarrow \mathcal{M}^{(1)} \leftarrow \cdots \leftarrow \mathcal{M}^{(4)}$ connecting various moduli spaces carracterized by different Hodge numbers:

$$\begin{gathered}
\left[\begin{array}{c||cccc} 7 & 2 & 2 & 2 & 2 \end{array}\right]^{1,65}_{-128} \leftarrow \left[\begin{array}{c||ccccc} 7 & 1 & 1 & 2 & 2 & 2 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{array}\right]^{2,58}_{-112} \leftarrow \left[\begin{array}{c||cccccc} 7 & 1 & 1 & 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right]^{3,51}_{-96} \\
\leftarrow \left[\begin{array}{c||ccccccc} 7 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right]^{4,44}_{-80} \leftarrow \left[\begin{array}{c||cccccccc} 7 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right]^{5,37}_{-64} \rightarrow \left[\begin{array}{c||c} 1 & 2 \\ 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{array}\right]^{4,68}_{-128} .
\end{gathered} \tag{3.10}$$

In this sequence, the last moduli space we denote $\mathcal{M}^{(1111)}$ was obtained by sending the volume v_7 of $\mathbb{CP}^7$ to zero. It is again obtained by determinantal contraction as follows: write the system of 8 equations associated to $\mathcal{M}^{(4)}$. Since these equations are linear in x_i $(i = 1, \ldots, 8)$ and that projective coordinates in $\mathbb{CP}^7$ cannot vanish simultaneously, the 8×8 determinant of coefficients of the x_i's must be zero. Denoting by y_j, z_j, t_j and u_j $(j = 1, 2)$ the projective coordinates of the 4 $\mathbb{CP}^1$'s used in the definition of $\mathcal{M}^{(4)}$, this determinant $\mathcal{D}^\sharp$ is an homogeneous polynomial of degree 2 in each of the variables y, z, t and u. When the volume of $\mathbb{CP}^7$ is shrunk to zero, we may integrate out the variables x_i and replace the 8 equations by the single quadratic equation

$$\mathcal{D}^\sharp(y, z, t, u) = 0\,. \tag{3.11}$$

This defines as in the previous section a singular variety in $\mathcal{M}^{(1111)}$ that can be deformed to a generic smooth manifold.

Actually, the moduli space $\mathcal{M}^{(1111)}$ plays a central role in the construction of the web of CY's since *any* CICY moduli space can be related to it by performing a

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finite number of determinantal splittings and contractions. We conclude this section by giving another example of sequence we shall consider later, originally considered in [1]. This sequence looks very similar to the one above but will yield different patterns for its G_2 descendants:

$$
\begin{aligned}
&\left[\begin{array}{c||cc} 5 & 4 & 2 \end{array}\right]^{1,89}_{-176} \leftarrow \left[\begin{array}{c||ccc} 5 & 4 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{array}\right]^{2,86}_{-168} \leftarrow \left[\begin{array}{c||cccc} 5 & 3 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \end{array}\right]^{3,69}_{-132} \\
&\leftarrow \left[\begin{array}{c||ccccc} 5 & 2 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{array}\right]^{4,46}_{-84} \leftarrow \left[\begin{array}{c||cccccc} 5 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right]^{5,37}_{-64} \rightarrow \left[\begin{array}{c||c} 1 & 2 \\ 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{array}\right]^{4,68}_{-128} .
\end{aligned}
\tag{3.12}
$$

3.3 Black hole resolution of the conifold singularity

The presence of singularities on the Kähler (resp. complex structure) moduli space of $(2,2)$ superconformal Calabi-Yau sigma-models where 2-cycles (resp. 3-cycles) vanish has been a long standing problem in the context of type II superstring compactifications on a CY threefold $\mathcal{C}$. The issue is particularly sharp in the case of the type IIB string at a complex structure singularity, since the metric on the vector multiplets is exact at tree-level in perturbation theory, and receives no worldsheet instanton corrections. The logarithmic singularity in the $\mathcal{N} = 2$ prepotential signals that some light degrees of freedom have been integrated out. Indeed, the type IIB theory possesses D3-branes, which by wrapping the vanishing 3-cycle yield light black hole states in four dimensions. Those are BPS hypermultiplets charged under the U(1) vector associated to the cycle and whose mass is proportional to the volume of the vanishing cycle, $m \propto V_{\gamma_3}/g_s$, where g_s is the string coupling constant. These states are massless at the conifold even at arbitrarily weak coupling and, when taken into account, yield a smooth low energy effective action [6]. The case of singularities in the Kähler structure of type IIA compactifications is identical, with D2-branes wrapped on the vanishing 2-cycle playing the role of the massless black holes, and we shall refer to these two cases as the "vector-conifold". The reversed process, namely singularities in the Kähler structure of type IIB compactifications, or complex structure of type IIA, occur in the hypermultiplet moduli space, and is of a different nature from physical point of view. We shall refer to them as the "hyper-conifold". In that case, the Euclidean D-string wrapped on the vanishing 2-cycle (or D2-brane wrapped on the vanishing 3-cycle) yield space-time instantons, that have been argued to correct the singular metric on the hypermultiplet moduli space [34]. At the same time, the wrapped D3 (or D4) yield tensionless strings, which provide the missing degrees of freedom [35]. Note that from the mathematical point of view,

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the vector- and hyper-conifold are the two sides of the conifold transition, since a two-cycle is shrunk to zero size and reappears as a three-cycle.

The mathematical transition between Calabi-Yau manifolds can be understood as Higgsing/un-Higgsing of the low-energy degrees of freedom as follows [7] (we phrase our discussion in terms of the type IIA vector-conifold). Consider a singularity in the Kähler moduli space, where N 2-cycles γ_a ($a = 1,\ldots,N$) go to zero size simultaneously ($N = 8$ in our first example eq. (3.6)). In general, these distinct cycles are not independent in homology, but they satisfy R relations of the form

$$\alpha_1^r\gamma_1 + \cdots + \alpha_N^r\gamma_N = 0 \qquad (r = 1,\ldots,R)\,, \tag{3.13}$$

for some integer α_a^r ($a = 1,\ldots,N; r = 1,\ldots,R$). The membranes wrapped around the 2-cycles give N black holes hypermultiplets, which are charged under the $(N-R)$ independent U(1) massless vector fields arising from the reduction of the 10-dimensional three-form C on the $(N-R)$ homology classes. In all the transitions considered in the previous section, the N vanishing cycles are all proportional in homology to the same S^2, whose volume is sent to zero, so that $R = N-1$.

Due to their charges, the hypermultiplets are no longer decoupled from the vectors, but couple in a way consistent with $\mathcal{N} = 2$ SUSY. In $\mathcal{N} = 1$ terminology, there is a superpotential

$$\mathcal{W} = \sum_{I=1}^{N-R}\sum_{a=1}^{N} q_I^a T^I h_a \tilde{h}_a \tag{3.14}$$

together with D-terms for each of the generators of the gauge group,

$$D_I = \sum_{a=1}^{N} q_I^a \left(|h_a|^2 - |\tilde{h}_a|^2\right), \tag{3.15}$$

where T^I are the complex scalar fields in the vector multiplets, and $(h_a, \tilde{h}_a)$ the chiral fields associated to the N black holes hypermultiplets, with charge q_I^a under the I-th U(1) factor. In the Coulomb phase where T^I condense, all hypermultiplets get a mass, yielding the massless spectrum on the CY $\mathcal{C}$. In the Higgs phase on the other hand, $3(N-R)$ among the $4N$ real scalars degrees of freedom are fixed by the D-term and T^I's F-term conditions, and another $(N-R)$ is gauged away by the $U(1)^{N-R}$ vector fields getting massive. This leaves $4R$ real flat directions in the potential corresponding to R neutral hypermultiplets. The spectrum in the Higgs phase consists of $h_{12}+R$ neutral hypermultiplets together with $h_{11}-(N-R)$ abelian vector multiplets coupled to $\mathcal{N} = 2$ supergravity. This is the spectrum of a compactification on a new CY $\mathcal{C}'$, whose Hodge numbers are $h'_{12} := h_{12}+R$ and $h'_{11} := h_{11}-(N-R)$. This is precisely what we found by determinantal contraction in all the examples considered in the previous section, as can be checked by using eq. (3.9).

4. Conifold transitions between G_2 manifolds

Having recalled the necessary background on the conifold transition between Calabi-Yau manifolds, we now discuss the transition between G_2 manifolds constructed by a *freely acting* quotient $(\mathcal{C} \times S^1)/\sigma$. The non-freely acting case will be discussed in section 5. We start by considering the same transition as in eq. (3.6), but now keeping track of the antiholomorphic involution σ.

4.1 An example

Let us consider a G_2 manifold constructed from our first example (3.1). Since the embeding space in this case is $\mathbb{CP}^7 \times \mathbb{CP}^1$, we may choose an involution of type A or B on each factor, thus giving 4 different orbifolds. In order to exclude the possibility of fixed points, we choose the involution B acting on $\mathbb{CP}^7$. As we shall see, this action can be combined with any of the involutions A or B on $\mathbb{CP}^1$ (and also involution C acting on two $\mathbb{CP}^1$'s), to give a consistent involution σ.

4.1.1 Involution $B \times B$ on $\mathbb{CP}^7 \times \mathbb{CP}^1$

We start by considering the configuration matrix

$$\mathcal{N}^{(1)} = \left[\begin{array}{c\|ccccc} \widehat{7} & 1 & 1 & 2 & 2 & 2 \\ \widehat{1} & 1 & 1 & 0 & 0 & 0 \end{array}\right]^{0,61} , \tag{4.1}$$

where we anticipated the values of b_2 and b_3 that will be determined momentarily. The invariance under the involution w: $x_i \to \bar{\hat{x}}_i$ $(i = 1, \ldots, 8)$, $y_j \to \bar{\hat{y}}_j$ $(j = 1, 2)$ puts constraints on the polynomials appearing in the system $(\mathcal{S}_1)$ of (3.2). Indeed, the transformed system under the involution should be equivalent to the complex conjugate of the original one:

$$\begin{cases} f_1(\bar{\hat{x}}, \bar{\hat{y}}) = f_2(\bar{\hat{x}}, \bar{\hat{y}}) = 0 \\ e_2(\bar{\hat{x}}) = e_3(\bar{\hat{x}}) = e_4(\bar{\hat{x}}) = 0 \end{cases} \iff \begin{cases} \bar{f}_1(\bar{x}, \bar{y}) = \bar{f}_2(\bar{x}, \bar{y}) = 0 \\ \bar{e}_2(\bar{x}) = \bar{e}_3(\bar{x}) = \bar{e}_4(\bar{x}) = 0\,, \end{cases} \tag{4.2}$$

where $\bar{f}_i(u,v)$ $(i = 1,2)$ is the polynomial $f_i(u,v)$, whose coefficients have been changed to their complex conjugates, while $\hat{x}_{2p-1} = -x_{2p}$, $\hat{x}_{2p} = x_{2p-1}$ $(p = 1,2,3,4)$, and similarly for $\bar{e}_n$ $(n = 2,3,4)$ and $\hat{y}_j$ $(j = 1,2)$. As a result, there should exist two matrices M and N in $GL(2,\mathbb{C})$ and $GL(3,\mathbb{C})$, respectively, such that

$$\begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix} = M \begin{pmatrix} \bar{f}_1(\hat{x},\hat{y}) \\ \bar{f}_2(\hat{x},\hat{y}) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} e_2(x) \\ e_3(x) \\ e_4(x) \end{pmatrix} = N \begin{pmatrix} \bar{e}_2(\hat{x}) \\ \bar{e}_3(\hat{x}) \\ \bar{e}_4(\hat{x}) \end{pmatrix} , \tag{4.3}$$

where we have used the fact that the dummy variables satisfy $\hat{\hat{x}} = -x$, $\hat{\hat{y}} = -y$. The consistency of this system imposes

$$MM^* = I \quad \text{and} \quad NN^* = I\,. \tag{4.4}$$

By considering linear changes of basis on the equations $f_{1,2}$ and $e_{2,3,4}$ similar to whose considered in section 2.3 for projective coordinates, it is possible to choose $M = I$, $N = I$, without loss of generality. As a result, the coefficients of the equations must satisfy

$$f_i(x, y) = \bar{f}_i(\hat{x}, \hat{y}) \quad (i = 1, 2) \quad \text{and} \quad e_j(x) = \bar{e}_j(\hat{x}) \qquad (j = 2, 3, 4) \tag{4.5}$$

or equivalently in terms of the polynomials P_{kl}:

$$P_{k1}(x) = \bar{P}_{k2}(\hat{x}) \qquad (k = 1, 2)\,. \tag{4.6}$$

This condition halves the number of parameters appearing in the coefficients of the defining polynomials. This is in agreement with the effect of the orbifold on the untwisted spectrum discussed in section 2.2, at least when the parameters appearing in the defining equations are in one-to-one correspondence with the complex structure moduli of the CY (see earlier footnote in section 3.1).

The Betti numbers of a G_2 manifold $\mathcal{G}_1$ belonging to $\mathcal{N}^{(1)}$ can be determined simply as follows. The harmonic $(1,1)$-forms of the CY are the pull-back of the Kähler forms J_7 and J_1 of the projective spaces $\mathbb{CP}^7$ and $\mathbb{CP}^1$, which are odd under the involution. Hence $h_{11}^+ = 0$, so that using $h_{11} = 2$, $h_{12} = 58$ for the CY, we find from (2.4) that $b_2 = 0$ and $b_3 = 1 + 2 + 58 = 61$. Note that this is the complete spectrum, since there are no fixed points that could contribute twisted sectors.

We can now perform the determinantal contraction described in section 3.1 by sending the volume v_1 of $\mathbb{CP}^1$ to zero and going to the description in terms of the system (3.4). We then note that the relations

$$e_1^\sharp(x) = \bar{e}_1^\sharp(\hat{x}) \quad \text{and} \quad e_j(x) = \bar{e}_j(\hat{x}) \quad (j = 2, 3, 4) \tag{4.7}$$

hold, thanks to eq. (4.6) and to the fact that P_{kl} are polynomials of odd degree. This defines a singular variety $\mathcal{G}_0^\sharp$ corresponding to a point in the G_2 moduli space, with configuration matrix

$$\mathcal{N}^{(0)} = \left[\widehat{7}\middle\| 2\ 2\ 2\ 2\right]^{0,67}. \tag{4.8}$$

Indeed, proceeding as for $\mathcal{N}^{(1)}$, it is easy to show that the 4 generic homogeneous equations of degree 2 in (3.8) have to satisfy the same constraints as in (4.7). $\mathcal{G}_0^\sharp$ may now be deformed into a smooth manifold on $\mathcal{N}^{(0)}$ by considering some real t and $\varepsilon^2(x)$ in (3.8) such that $\varepsilon^2(x) \equiv \bar{\varepsilon}^2(\hat{x})$. The Betti numbers on $\mathcal{N}^{(0)}$ are found in the same way as above: The Kähler form J_7 is odd, so that $h_{11}^+ = 0$, which along with $h_{11} = 1$, $h_{12} = 65$ implies $b_2 = 0$ and $b_3 = 67$. As a result, we have described the conifold transition obtained by determinantal contraction from $\mathcal{N}^{(1)}$ to $\mathcal{N}^{(0)}$. This will be denoted:

$$\left[\widehat{7}\middle\| 2\ 2\ 2\ 2\right]^{0,67} \leftarrow \left[\begin{array}{c|c} \widehat{7} & 1\ 1\ 2\ 2\ 2 \\ \widehat{1} & 1\ 1\ 0\ 0\ 0 \end{array}\right]^{0,61}. \tag{4.9}$$

On the double cover of the orbifold $\mathcal{G}_1$, at each point of S^1, the CY admitting the isometry $B \times B$ have 8 S^2's that have shrunk to points on $\mathcal{G}_0^\sharp$ before being deformed to S^3's on $\mathcal{G}_0$.

4.1.2 Involution $B \times A$ on $\mathbb{CP}^7 \times \mathbb{CP}^1$

We now consider the configuration matrix

$$\begin{bmatrix} \widehat{7} & 1\ 1\ 2\ 2\ 2 \\ \overline{1} & 1\ 1\ 0\ 0\ 0 \end{bmatrix}^{0,61} . \tag{4.10}$$

The system $(\mathcal{S}_1)$ of eq. (3.2) must satisfy

$$\begin{cases} f_1(\bar{\hat{x}}, \bar{y}) = f_2(\bar{\hat{x}}, \bar{y}) = 0 \\ e_2(\bar{\hat{x}}) = e_3(\bar{\hat{x}}) = e_4(\bar{\hat{x}}) = 0 \end{cases} \iff \begin{cases} \bar{f}_1(\bar{x}, \bar{y}) = \bar{f}_2(\bar{x}, \bar{y}) = 0 \\ \bar{e}_2(\bar{x}) = \bar{e}_3(\bar{x}) = \bar{e}_4(\bar{x}) = 0 \,, \end{cases} \tag{4.11}$$

which implies now that there exist two matrices M and N in $\mathrm{GL}(2, \mathbb{C})$ and $\mathrm{GL}(3, \mathbb{C})$, respectively, such that

$$\begin{pmatrix} f_1(x, y) \\ f_2(x, y) \end{pmatrix} = M \begin{pmatrix} \bar{f}_1(\hat{x}, y) \\ \bar{f}_2(\hat{x}, y) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} e_2(x) \\ e_3(x) \\ e_4(x) \end{pmatrix} = N \begin{pmatrix} \bar{e}_2(\hat{x}) \\ \bar{e}_3(\hat{x}) \\ \bar{e}_4(\hat{x}) \end{pmatrix} . \tag{4.12}$$

Since $f_i(x, y)$ $(i = 1, 2)$ is of odd degree in x, eq. (4.4) is now replaced by

$$MM^* = -I \quad \text{and} \quad NN^* = I \,. \tag{4.13}$$

By considering changes of basis in the defining equations, we may impose without loss of generality as in the previous section M=$\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $N = I$:

$$f_1(x, y) = \bar{f}_2(\hat{x}, y) \quad \text{and} \quad e_j(x) = \bar{e}_j(x) \quad (j = 2, 3, 4) \tag{4.14}$$

or equivalently

$$P_{1l}(x) = \bar{P}_{2l}(\hat{x}) \qquad (l = 1, 2) \,. \tag{4.15}$$

The Betti numbers of the G_2 orbifold can be computed in the same way as before, yielding $b_2 = 0$, $b_3 = 61$. Upon sending the volume of $\mathbb{CP}^1$ to zero, the system becomes $(\mathcal{S}_0^\sharp)$ in eq. (3.4), which satisfies eq. (4.7), as can be seen from eq. (4.15). This shows that we arrived at a singular point of the moduli space $\mathcal{N}^{(0)}$. As before, we can then deform the orbifold to obtain a smooth manifold of $\mathcal{N}^{(0)}$. Thus, we find another transition, whose end point is in $\mathcal{N}^{(0)}$:

$$\begin{bmatrix} \widehat{7} \| 2\ 2\ 2\ 2 \end{bmatrix}^{0,67} \leftarrow \begin{bmatrix} \widehat{7} & 1\ 1\ 2\ 2\ 2 \\ \overline{1} & 1\ 1\ 0\ 0\ 0 \end{bmatrix}^{0,61} . \tag{4.16}$$

4.1.3 Involution $B \times C$ on $\mathbb{CP}^7 \times \mathbb{CP}^1 \times \mathbb{CP}^1$

So far, all the G_2 manifolds that we constructed had $b_2 = 0$. This is because the Kähler forms $J_{7,1}$ were odd so that h_{11}^+ was always zero. We are going to see now that this is not a general feature, when one uses involutions C. As an example, let us consider the configuration matrix

$$\left[\begin{array}{c||cccccc} \widehat{7} & 1 & 1 & 1 & 1 & 2 & 2 \\ \overset{\frown}{1} & 1 & 1 & 0 & 0 & 0 & 0 \\ \underset{\smile}{1} & 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right]^{1,54} , \tag{4.17}$$

where b_2 will turn out to be one. The corresponding system of equations

$$(\mathcal{S}_2) \begin{cases} f_1(x,y) := P_{11}(x)y_1 + P_{12}(x)y_2 = 0 \\ f_2(x,y) := P_{21}(x)y_1 + P_{22}(x)y_2 = 0 \\ f_3(x,z) := P_{31}(x)z_1 + P_{32}(x)z_2 = 0 \\ f_4(x,z) := P_{41}(x)z_1 + P_{42}(x)z_2 = 0 \\ e_3(x) = e_4(x) = 0 \,, \end{cases} \tag{4.18}$$

where $P_{k,l}$ ($k = 1, \ldots, 4$; $l = 1, 2$) and e_n ($n = 3, 4$) are polynomials in x_i's of degree 1 and 2, respectively, while z_j ($j = 1, 2$) are projective coordinates for the second $\mathbb{CP}^1$ factor. For this system to define a manifold satisfying the discret isometry B on $\mathbb{CP}^7$, C on $\mathbb{CP}^1 \times \mathbb{CP}^1$, we must have

$$\begin{cases} f_1(\bar{\hat{x}},\bar{y}) = f_2(\bar{\hat{x}},\bar{y}) = 0 \\ f_3(\bar{\hat{x}},\bar{z}) = f_4(\bar{\hat{x}},\bar{z}) = 0 \\ e_3(\bar{\hat{x}}) = e_4(\bar{\hat{x}}) = 0 \,, \end{cases} \iff \begin{cases} \bar{f}_1(\bar{x},\bar{z}) = \bar{f}_2(\bar{x},\bar{z}) = 0 \\ \bar{f}_3(\bar{x},\bar{y}) = \bar{f}_4(\bar{x},\bar{y}) = 0 \\ \bar{e}_3(\bar{x}) = \bar{e}_4(\bar{x}) = 0 \,, \end{cases} \tag{4.19}$$

which shows that their exist M, M' and N in $GL(2,\mathbb{C})$ such that

$$\begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix} = M \begin{pmatrix} \bar{f}_3(\hat{x},y) \\ \bar{f}_4(\hat{x},y) \end{pmatrix}, \quad \begin{pmatrix} f_3(x,z) \\ f_4(x,z) \end{pmatrix} = M' \begin{pmatrix} \bar{f}_1(\hat{x},z) \\ \bar{f}_2(\hat{x},z) \end{pmatrix}$$

$$\text{and} \quad \begin{pmatrix} e_3(x) \\ e_4(x) \end{pmatrix} = N \begin{pmatrix} \bar{e}_3(\hat{x}) \\ \bar{e}_4(\hat{x}) \end{pmatrix}. \tag{4.20}$$

The consistency of this system implies that $MM'^* = -I$ and $NN^* = I$, so that we may choose without loss of generality $M = -M' = I, N = I$. As a result, the configuration matrix in eq. (4.17) corresponds to the system $(\mathcal{S}_2)$ in eq. (4.18), with the constraints

$$f_k(x,y) = \bar{f}_{k+2}(\hat{x},y) \quad (k = 1,2) \quad \text{and} \quad e_j(x) = \bar{e}_j(x) \quad (j = 3,4) \,, \tag{4.21}$$

which gives

$$P_{kl}(x) = \bar{P}_{k+2,l}(\hat{x}) \quad (k,l = 1,2) \,. \tag{4.22}$$

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To determine the Betti numbers, denoting by J_7, J_1, J_1' the Kähler forms of the three projective spaces, we note that J_7 and $J_1 + J_1'$ are still odd, but $J_1 - J_1'$ is even. The volume of the two $\mathbb{CP}^1$ are therefore restricted to be the same, but their common volume is free to vary. Hence $h_{11}^+ = 1$, and since $h_{11} = 3$, $h_{12} = 51$, we find $b_2 = 1$, $b_3 = 1 + 2 + 51 = 54$. These are the exact values of the Betti numbers, since again there are no fixed points that could contribute twisted sectors.

Let us now replace the equations $f_i(x, y) = 0$ $(i = 2, 4)$ in eq. (4.18) by the vanishing determinants

$$\begin{aligned} e_1^\sharp(x) &= P_{11}(x)P_{22}(x) - P_{21}(x)P_{12}(x) = 0 \\ e_2^\sharp(x) &= P_{31}(x)P_{42}(x) - P_{41}(x)P_{32}(x) = 0 \end{aligned} \tag{4.23}$$

and send the volume of both $\mathbb{CP}^1$ to zero, so that the system becomes

$$\begin{cases} e_1^\sharp(x) = e_2^\sharp(x) = 0 \\ e_3(x) = e_4(x) = 0\,. \end{cases} \tag{4.24}$$

We then note that thanks to eq. (4.22), we have

$$\begin{pmatrix} e_1^\sharp(x) \\ e_2^\sharp(x) \end{pmatrix} = N \begin{pmatrix} \bar{e}_1^\sharp(\hat{x}) \\ \bar{e}_2^\sharp(\hat{x}) \end{pmatrix}, \quad \text{where} \quad N = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.25}$$

Since $NN^* = I$, we could by a change of basis in the 2 dimensional vectorial space of equations $e_{1,2}^\sharp$ replace N by the identity matrix, as we did for $e_{3,4}$ in eq. (4.20). As a result, the equations in (4.24) define a singular space in $\mathcal{N}^{(0)}$ that we can deform to a smooth manifold by adding $t_i\varepsilon_i^2(x)$ $(i = 1, 2)$ to the right hand sides of the two first equations, where $t_{1,2}$ are real numbers and $\varepsilon_{1,2}^2$ are generic homogeneous polynomials of degree two satisfying $\varepsilon_1^2(x) = \bar{\varepsilon}_2^2(\hat{x})$. This is summarized as

$$\left[\begin{array}{c||cccc} \widehat{7} & 2 & 2 & 2 & 2 \end{array}\right]^{0,67} \leftarrow \left[\begin{array}{c||cccccc} \widehat{7} & 1 & 1 & 1 & 1 & 2 & 2 \\ \overset{\frown}{1} & 1 & 1 & 0 & 0 & 0 & 0 \\ \underset{\smile}{1} & 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right]^{1,54}. \tag{4.26}$$

4.2 The web of G_2 manifolds (CICY $\times\, S^1)/\sigma$

As we have seen in section 3.2, all complete intersection CY moduli spaces are connected to $\mathcal{M}^{(1111)}$, the moduli space associated to the last configuration matrix in eq. (3.10). As a result, G_2 manifolds constructed by orbifolding a product $\mathcal{C} \times S^1$, where $\mathcal{C}$ is a CICY may also be connected to one of the G_2 manifolds descending from $\mathcal{M}^{(1111)}$. There are 9 possible choices of antiholomorphic involutions, 6 of them involve at least one involution B on a projective factor and are freely acting:

$$\left[\begin{array}{c||c} \widehat{1} & 2 \\ \widehat{1} & 2 \\ \widehat{1} & 2 \\ \widehat{1} & 2 \end{array}\right]^{0,73}, \left[\begin{array}{c||c} \widehat{1} & 2 \\ \widehat{1} & 2 \\ \widehat{1} & 2 \\ \overline{1} & 2 \end{array}\right]^{0,73}, \left[\begin{array}{c||c} \widehat{1} & 2 \\ \widehat{1} & 2 \\ \overline{1} & 2 \\ \overline{1} & 2 \end{array}\right]^{0,73}, \left[\begin{array}{c||c} \widehat{1} & 2 \\ \overline{1} & 2 \\ \overline{1} & 2 \\ \overline{1} & 2 \end{array}\right]^{0,73}, \left[\begin{array}{c||c} \widehat{1} & 2 \\ \widehat{1} & 2 \\ \overset{\frown}{1} & 2 \\ \underset{\smile}{1} & 2 \end{array}\right]^{1,72}, \left[\begin{array}{c||c} \widehat{1} & 2 \\ \overline{1} & 2 \\ \overset{\frown}{1} & 2 \\ \underset{\smile}{1} & 2 \end{array}\right]^{1,72}, \tag{4.27}$$

while the last 3 have fixed points,

$$\left[\begin{array}{c||c} \bar{1} & 2 \\ \bar{1} & 2 \\ \bar{1} & 2 \\ \bar{1} & 2 \end{array}\right]^{0,73} , \left[\begin{array}{c||c} \bar{1} & 2 \\ \bar{1} & 2 \\ \overset{\frown}{1} & 2 \\ \underset{\smile}{1} & 2 \end{array}\right]^{1,72} , \left[\begin{array}{c||c} \overset{\frown}{1} & 2 \\ \underset{\smile}{1} & 2 \\ \overset{\frown}{1} & 2 \\ \underset{\smile}{1} & 2 \end{array}\right]^{2,71} . \tag{4.28}$$

Here the indicated Betti numbers do not take into account the contributions from the twisted sectors. This is not to say that the web of G_2 manifolds descending from CICY splits into 9 disconnected components. Indeed, it is easy to find sequences of determinantal splittings and contractions that relate any of these 9 cases, as shown in figure 1.

In the sequences we displayed, the absence of fixed points is ensured by taking an involution of type B on the $\mathbb{CP}^7$ factor. This rule is broken only when arriving to the three non-freely acting configurations in eq. (4.28), shown with dotted arrows on figure 1. It would be interesting to understand better these transitions with fixed points. For illustrative purposes, we also give in figure 2 the sequences of G_2 manifolds descendings the CY sequence (3.12) using the involution B on $\mathbb{CP}^5$, so that the involution is freely acting. It is easy to check that the sequences shown are the only ones for which the involution is compatible with the CICY matrix, as discussed in section 2.4. Although the above sequences are suggestive, we have not shown that all G_2 manifolds $(\mathrm{CICY}\times S^1)/\sigma$ could be related to one of the 9 configuration matrices above, since intermediate steps could in principle involve non-freely acting configurations.

Finally, we note that several of the manifolds in figure 1 have the same value of $b_2 + b_3$. As we mentioned in the introduction, this is a necessary condition for G_2 manifolds to be mirror to each other, and it would be interesting to check if this is indeed the case.

4.3 Black hole condensation in $\mathcal{N} = 1$ vacua

Having described conifold transitions between complete intersection G_2 manifolds at the mathematical level, we now would like to understand these processes in string or field theory terms. As long as the nodal points are not fixed under the involution, the physical mechanisms will be very similar to the standard conifold case for CY manifolds in $\mathcal{N} = 2$ type II constructions. In [36], $\mathbb{Z}_2$ orbifolds of M-theory on $K_3 \times S^1$ were considered, breaking $\mathcal{N} = 2$ to $\mathcal{N} = 1$ in 6 dimensions: In some cases, even though the additional states arising at specific points in moduli space were not BPS any more, they still inherited their mass formula from the $\mathcal{N} = 2$ theory, so that they were still massless after orbifolding. Similar arguments apply in four dimensions. The precise mechanism will actually turn out to be somewhat different depending whether b_2 remains constant or decreases.

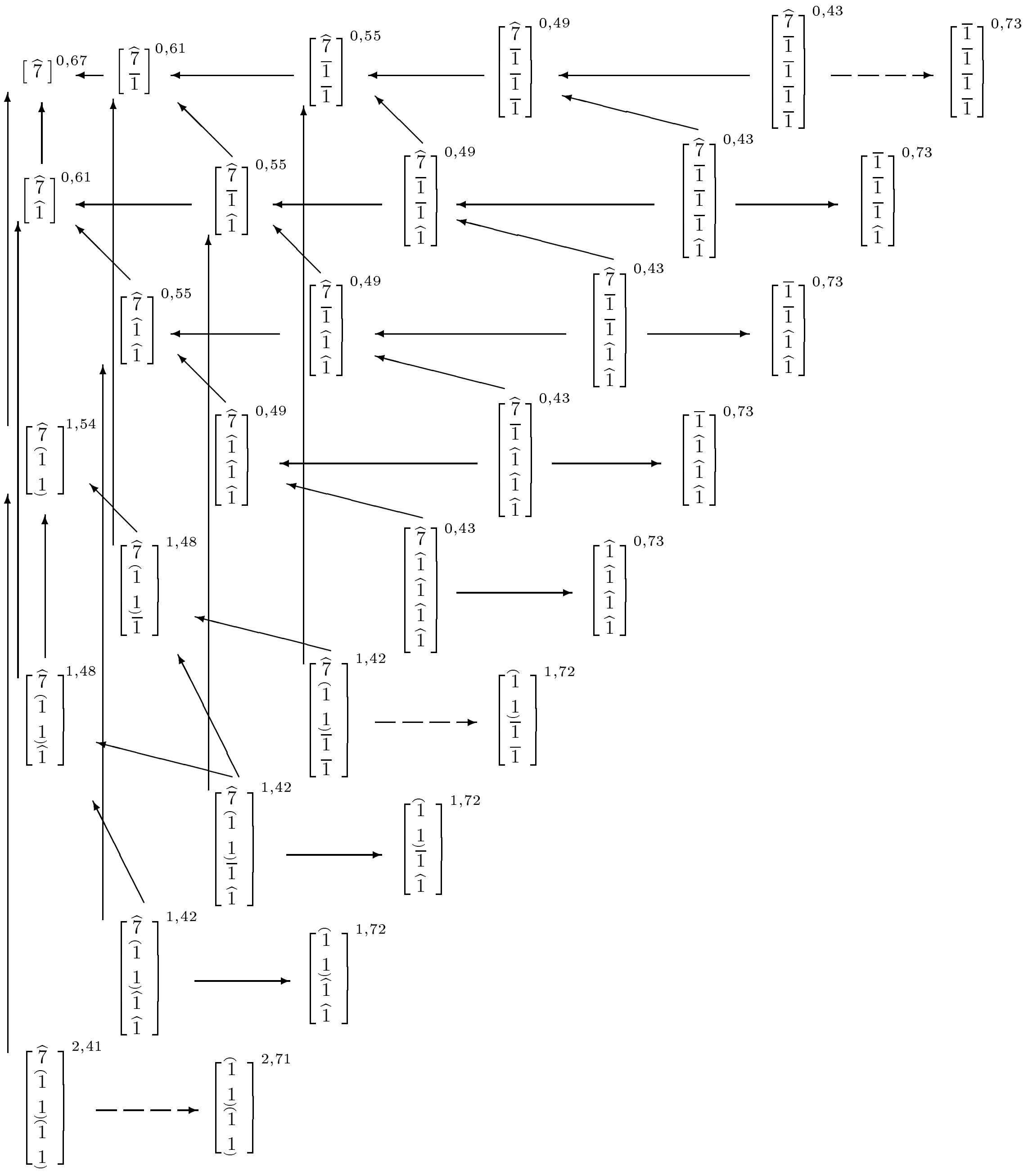

Figure 1: Sequences of transitions relating the 9 G_2 configuration matrices descending from $\mathcal{M}^{(1111)}$. Only the first column of the G_2 configuration matrices is shown, the other ones being identical to those appearing in the CY configuration matrices of eq. (3.10). Dotted arrows denote transitions to non-freely acting orbifolds.

$$
\begin{array}{ccccccccccc}
 & \swarrow & \begin{bmatrix} \widehat{5} \\ \overline{1} \end{bmatrix}^{0,89} & \leftarrow & \begin{bmatrix} \widehat{5} \\ \widehat{1} \\ \overline{1} \end{bmatrix}^{0,73} & \leftarrow & \begin{bmatrix} \widehat{5} \\ \overline{1} \\ \widehat{1} \\ \overline{1} \end{bmatrix}^{0,51} & \leftarrow & \begin{bmatrix} \widehat{5} \\ \widehat{1} \\ \overline{1} \\ \widehat{1} \\ \overline{1} \end{bmatrix}^{0,43} & \rightarrow & \begin{bmatrix} \widehat{1} \\ \overline{1} \\ \widehat{1} \\ \overline{1} \end{bmatrix}^{0,73} \\
\begin{bmatrix} \widehat{5} \end{bmatrix}^{0,91} & & & & & & & & & & \\
 & \nwarrow & \begin{bmatrix} \widehat{5} \\ \widehat{1} \end{bmatrix}^{0,89} & \leftarrow & \begin{bmatrix} \widehat{5} \\ \widehat{1} \\ \widehat{1} \end{bmatrix}^{0,73} & \leftarrow & \begin{bmatrix} \widehat{5} \\ \overline{1} \\ \widehat{1} \\ \widehat{1} \end{bmatrix}^{0,51} & \leftarrow & \begin{bmatrix} \widehat{5} \\ \widehat{1} \\ \overline{1} \\ \widehat{1} \\ \widehat{1} \end{bmatrix}^{0,43} & \rightarrow & \begin{bmatrix} \widehat{1} \\ \overline{1} \\ \widehat{1} \\ \widehat{1} \end{bmatrix}^{0,73} .
\end{array}
$$

Figure 2: Sequences of transitions descending from the CICY sequence (3.12). Only the first column of the matrices is shown, the other are as in (3.12).

4.3.1 Transition at constant b_2

This situation arises when the $(N-R)$ homology classes of the N 2-cycles vanishing at the singularity of the CY are odd under the involution w. Accordingly, the $(N-R)$ U(1) gauge fields present in the $\mathcal{N}=2$ case are projected out, while the volumes of the cycles together with the B-fluxes remain as chiral fields of the $\mathcal{N}=1$ theory. As the 2-cycles shrink to zero size and grow into 3-cycles, no extra gauge fields appear, so that b_2 remains constant.

Since by assumption the antiholomorphic involution has no fixed point, it maps each node a to a different node $w(a)$ (in particular, the number of nodes N has to be even). Since for the cycles $\gamma_a \to -\gamma_{w(a)}$, the Ramond-Ramond charged black hole fields have to satisfy the projection relation

$$
h_a(x) = \tilde{h}_{w(a)}(-x)\,, \qquad \tilde{h}_a(x) = h_{w(a)}(-x)\,, \tag{4.29}
$$

where x is the coordinate along S^1. Ordering the N nodal points so that $w(a) = a + N/2$, the even combinations $H_a = h_a + \tilde{h}_{a+N/2}$ and $\tilde{H}_a = \tilde{h}_a + h_{a+N/2}$ therefore remain massless, while the odd combinations $h_a - \tilde{h}_{a+N/2}$ and $\tilde{h}_a - h_{a+N/2}$ are projected out, or rather acquire a Kaluza-Klein mass $1/2R$. The superpotential (3.14) therefore reduces to

$$
\mathcal{W} = \sum_{I=1}^{N-R} \sum_{a=1}^{N/2} q_I^a T^I H_a \tilde{H}_a\,, \tag{4.30}
$$

with usual diagonal quadratic kinetic terms. In the phase where the complex scalar fields T^I condense, all black hole fields $H_a, \tilde{H}_a$ acquire a mass, and we find the massless spectrum corresponding to the original G_2 manifold $\mathcal{G}$. If on the other hand the black hole fields $H_a, \tilde{H}_a$ condense, the $(N-R)$ chiral fields T^I acquire a mass.

The counting of the massless spectrum goes as follows: To the b_3 chiral multiplets at a generic point of the moduli space, we add N ones associated to the $H_a, \tilde{H}_a$, substract $(N-R)$ from F-terms associated to each of the T^I's and another $(N-R)$ from the T^I that become massive. As a result, we find a new branch of flat directions of complex dimension $b_3 + 2R - N$. This spectrum corresponds to a compactification on a G_2 manifold, whose Betti numbers b'_2, b'_3 satisfy

$$b'_2 = b_2 \quad \text{and} \quad b'_3 = b_3 + 2R - N\,. \tag{4.31}$$

In the examples we considered explicitly, since we had $R = N-1$, we have $b'_3 = b_3 + N - 2$, reproducing the Betti numbers of the examples we considered with b_2 constant, as can be checked using eq. (3.9). The transition between G_2 manifolds at constant b_2 is therefore realized physically by the transition between two branches of $\mathcal{N}=1$ vacua.

4.3.2 Transition at decreasing b_2

This situation occurred in the examples we considered when some $\mathbb{CP}^n \times (\mathbb{CP}^n)'$ pairs of factors acted upon by involutions C were shrunk to zero size. Let N be the number of nodal points arising from the vanishing of the Kähler moduli of $\mathbb{CP}^n$. Under the involution w, these are exchanged with the N nodal points arising from the vanishing of the other projective factor. By the same token as above, the $2N$ black hole hypermultiplets of the $\mathcal{N}=2$ theory reduce to $2N$ chiral multiplets $H_a, \tilde{H}_a$. On the other hand, the $2(N-R)$ $\mathcal{N}=2$ vector multiplets reduce to $(N-R)$ $\mathcal{N}=1$ vector multiplets (from the even homology) and $(N-R)$ chiral multiplets (from the odd homology). The latter are neutral under the gauge group $\mathrm{U}(1)^{N-R}$, while the black hole states have charge $\pm q^I_a$. The scalars in this theory interact through the $\mathcal{N}=1$ superpotential

$$\mathcal{W} = \sum_{a=1}^{N} \sum_{I=1}^{N-R} q^I_a T^I H_a \tilde{H}_a\,, \tag{4.32}$$

and the D-terms

$$D^I = \sum_{a=1}^{N} q^I_a \left(|H_a|^2 - |\tilde{H}_a|^2 \right). \tag{4.33}$$

In the Coulomb phase, where the T^I condense and give a mass to the charged scalars $H_a, \tilde{H}_a$, we find the expected massless spectrum of the compactification on the G_2 manifold $\mathcal{G}$. In the Higgs phase, where $H_a, \tilde{H}_a$ acquire an expectation value, the gauge group is Higgsed and the T^I acquire a mass. The counting of massless chiral fields goes as follows: We start with $b_3 + 2N$ chiral fields, impose $(N-R)$ F-term conditions associated to each of the T^I's as well as $(N-R)$ real D-term conditions, gauge fix $(N-R)$ real broken generators and finally give a mass to the $(N-R)$ T^I fields. The resulting spectrum is therefore that of a compactification on a new G_2 manifold $\mathcal{G}'$ with Betti numbers

$$b_2' = b_2 - (N-R) \quad \text{and} \quad b_3' = b_3 + 3R - N\,. \tag{4.34}$$

These Betti numbers are precisely what we found from determinantal contraction in the cases where involutions C were considered, as can be checked from eq. (3.9), renaming N into $2N$ and taking $R = N-1$. We see that this situation is similar to what was already happening in the $\mathcal{N}=2$ case, namely the transition from one moduli space to another is a realization of a Higgs mechanism.

5. Fixed points and conifolds

Our discussion of conifold transitions in G_2 manifolds has so far been restricted to the case where the antiholomorphic involution acts without fixed points. A simple way to achieve this was to choose the freely-acting involution B for one of the factors of the projective space. Clearly, even if the involution has fixed points, the mechanism of the conifold transitions in G_2 manifolds will remain the same as long as the nodal point themselves are not fixed under the involution, but exchanged with one another. On the other hand, when a nodal point is fixed by the antiholomorphic involution, the local geometry changes drastically, and new phenomena can be expected. Here we will content ourselves with a simple example, leaving a more thorough inverstigation to future work.

Let us consider the local conifold geometry

$$z_1^2 + z_2^2 + z_3^2 + z_4^2 = \epsilon \tag{5.1}$$

and the type A involution w: $z_i \to \bar{z}_i$, requiring $\epsilon \in \mathbb{R}$. Writing $z_i = x^i + iy_i$, we recognize the defining equations $(x^i)^2 - y_i^2 = \epsilon$, $x^i y_i = 0$ for the cotangent bundle T^*S^3. For $\epsilon < 0$, there is no real solution to equation (5.1), so that the orbifold $(T^*S^3 \times \mathbb{R})/w\mathcal{I}$, where $\mathcal{I}$: $x \to -x$ on $\mathbb{R}$ defines a smooth non-compact G_2 manifold. For $\epsilon > 0$ on the other hand, there is a non-empty fixed point set, namely the zero section S^3 of the bundle T^*S^3. The quotient $(T^*S^3 \times \mathbb{R})/w\mathcal{I}$ now has a conical singularity, locally $S^3 \times \mathbb{R}^4/\mathbb{Z}_2$. Since $b_1(S^3) = 0$, the singularity cannot be resolved so as to preserve G_2 holonomy. The moduli space is therefore restricted to $\epsilon \le 0$, with a SU(2) enhanced symmetry point at $\epsilon = 0$ coming from membranes wrapping the $\mathbb{R}^4/\mathbb{Z}_2$ singularity. At $\epsilon = 0$, the collapsed 3-cycle may be grown up into a 2-cycle, changing the topology of the Calabi-Yau threefold to an $\mathrm{O}(-1) \times \mathrm{O}(-1)$ bundle over S^2, described by

$$\begin{pmatrix} z_1 + iz_2 & z_3 + iz_4 \\ -z_3 + iz_4 & z_1 - iz_2 \end{pmatrix} \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} = 0\,. \tag{5.2}$$

The involution w now has to act as $(\zeta_1, \zeta_2) \to (-\bar{\zeta}_2, \bar{\zeta}_1)$ in order to leave this equation invariant, and therefore is freely acting. We therefore have a smooth G_2 manifold on this side as well. It would be interesting to understand this phase structure from the point of view of the SU(2) gauge theory living at the singularity. In particular, the absence of G_2 resolution at $\epsilon > 0$ may correspond to spontaneous supersymmetry breaking, and realize a dual version of the scenario proposed in [37] in the context of $\mathcal{N} = 1$ theories on type A branes. More generally, one may consider cases where the special Lagrangian 3-cycle of fixed points undergoes transitions of the type considered by Joyce [38], so that the resulting G_2 manifold experiences a topology change.

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